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Integral formalism for surface electromagnetic waves in bianisotropic media

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Abstract

An integral approach is presented in the theory of surface electromagnetic waves propagating along the plane interface of bianisotropic non-absorbing media including optically active gyrotropic and bigyrotropic ones. This approach gives a uniform way of obtaining the dispersion equation for surface polaritons for an arbitrary cut section of the bianisotropic crystals and allows us to establish the existence conditions of surface polaritons. An example of application of this approach for the boundary of bianisotropic and isotropic media is given.

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1. Introduction

In recent years optical properties of new non-traditional materials such as photonic crystals, composites, etc have been widely investigated [1–3]. Light propagation in anisotropic gyrotropic and bianisotropic media is of interest as well. Optical effects in such media, which are not observable in the usual isotropic materials, can be used for construction of new or improved opto-electronic devices. Material anisotropy and bianisotropy under certain conditions can noticeably affect characteristics of both body and surface electromagnetic waves.

It is known that surface electromagnetic waves (surface polaritons) can exist in isotropic media when one of these has a negative dielectric permittivity. Such surface excitations are well studied [4–6]. As a rule the dielectric permittivity is negative when materials are excited near critical frequencies and strong dispersion takes place. However in papers [7, 8] it was shown that surface modes of a fundamentally new type are possible in media with positive dielectric permittivities and small frequency dispersion. It is important that the existence of such waves results from anisotropy in at least one of the contacting media. In addition dielectric permittivities should be chosen carefully, for instance $0 < \varepsilon_{\mathrm{o}} < \varepsilon' < \varepsilon_{\mathrm{e}}$ for the interface of a uniaxial crystal with permittivities $\varepsilon_{\mathrm{o}}, \varepsilon_{\mathrm{e}}$ and isotropic medium with permittivity ε'.

A specific feature of such surface polaritons is their inability to propagate in certain directions along the interface. The range of the allowed propagation directions is represented by sectors coplanar to the interface. It is significant that the higher the degree of anisotropy of the contacting media, the larger the angular width of the sectors of the allowed propagation directions for surface polaritons.

Taking into account medium anisotropy considerably complicates the bulk of calculations in the theory of surface waves. For surface acoustical waves in anisotropic media, Barnett and Lothe [9, 10] developed a special integral approach. They use it to obtain some qualitative results (existence theorems), which allow us to establish the possibility of surface wave propagation along any given direction at the interface and for any cut section of the crystal. Such an approach is also applicable to the theory of surface polaritons. For non-magnetic non-gyrotropic anisotropic media, the theory has been developed in papers [11, 12].

The purpose of the present paper is to extend the integral approach to bianisotropic media including optically active gyrotropic and bigyrotropic ones [13]. Surface polaritons of microwave and infra-red spectral bands in such media are an object of intensive theoretical and experimental studies [14, 15]. In our paper the system of wave equations in Stroh form [16] is obtained for the first time for homogeneous and inhomogeneous waves in stratified bianisotropic media. Integral representation for the surface impedance tensors of contacting media is then derived. With the use of these tensors, one can uniformly derive the dispersion equations for surface polaritons at an arbitrary cut section of the contacting media. These equations are represented in the form $F(\omega/ck) = 0$ with a monotonic function F, and existence of their solutions can be analysed comparatively simply. Note that the algorithm proposed by us for derivation of the dispersion equations can be programmed analytically (using the systems of computer algebra) with the material tensors of media as input data.

The paper is organized as follows. In section 2 we consider wave propagation in a stratified one-dimensional bianisotropic medium. Starting from Maxwell equations we derive the system of the first-order differential equations for vectors $\binom{\boldsymbol{H}_\tau}{\boldsymbol{q}\times\boldsymbol{E}}$. There are different forms of representation of such equations [17, 18]. We obtain the system of equations in Stroh form [16] with the use of tensorial bilinear forms $(\boldsymbol{uv})$ with two vector arguments. It appears that the equations obtained are convenient for further analysis of surface wave propagation. In section 3 two procedures for deriving the dispersion equation for surface electromagnetic waves are presented. The first one is conventional and based on solving the Fresnel equation and boundary conditions. The second one is based on the surface impedance tensor formalism. In section 4 integral representation of the surface impedance tensors is given on the basis of eigenvalue and eigenvector analysis for the system matrix obtained in section 2. Finally, in section 5 with the use of the presented integral formalism we consider an example of the dispersion equation derivation for surface waves at the interface of isotropic medium and bianisotropic crystals with symmetry 3 m, 4 mm or 6 mm. Crystals of these symmetry classes are described by constitutive equations more complicated than those for bi-isotropic media but simpler than those for bianisotropic crystals of other symmetry classes.

During derivation of the main formulae we use Fedorov's covariant methods [19, 20], and the following notation of operations with scalars, vectors and tensors are used. The scalar product of the vectors is marked as $\boldsymbol{ab}$, vector product as $\boldsymbol{a}\times\boldsymbol{b}$ and tensor product (also termed dyadic product or dyad) as $\boldsymbol{a}\otimes\boldsymbol{b}$. In some formulae the operation of scalar by vector multiplication is marked by a dot. For example, the form $\boldsymbol{a}\cdot\boldsymbol{bc}$ means that vector $\boldsymbol{a}$ is multiplied by a quantity, which equals the scalar product of vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

2. Wave equations in Stroh form for bianisotropic media

Propagation of the monochromatic electromagnetic waves in inhomogeneous bianisotropic media is described by Maxwell's equations

$$\nabla^\times \boldsymbol{E} = \frac{\mathrm{i}\omega}{c}\boldsymbol{B} \qquad \nabla^\times \boldsymbol{H} = -\frac{\mathrm{i}\omega}{c}\boldsymbol{D} \tag{1}$$

and by constitutive equations

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} + \alpha\boldsymbol{H} \qquad \boldsymbol{B} = \beta\boldsymbol{E} + \mu\boldsymbol{H} \tag{2}$$

or in an equivalent form

$$\boldsymbol{E} = \varepsilon^{-1}\boldsymbol{D} - \hat{\alpha}\boldsymbol{H} \qquad \boldsymbol{B} = \hat{\beta}\boldsymbol{D} + \hat{\mu}\boldsymbol{H} \tag{3}$$

where $\hat{\alpha} = \varepsilon^{-1}\alpha$, $\hat{\beta} = \beta\varepsilon^{-1}$, $\hat{\mu} = \mu - \beta\varepsilon^{-1}\alpha$.

Tensors $\varepsilon = \varepsilon(\omega)$, $\mu = \mu(\omega)$, $\alpha = \alpha(\omega)$, $\beta = \beta(\omega)$ in equations (2) depend on the wave frequency ω and the position vector $\boldsymbol{r}$.

For non-absorbing bianisotropic media $\varepsilon^+ = \varepsilon$, $\mu^+ = \mu$, $\beta^+ = \alpha$, where the superscript + marks the Hermitian conjugate operation. In equations (1) $\nabla^\times$ is the antisymmetric tensor called a dual to the vector ∇ (for any vector $\boldsymbol{v}$ the dual tensor equals $(\boldsymbol{v}^\times)_{ik} = e_{ijk}v_j$ [19, 20] where e_{ijk} is the Levi-Civita pseudotensor).

We consider stratified bianisotropic medium, for which the tensors ε, μ, α and β depend on spatial coordinate $z = \boldsymbol{q}\boldsymbol{r}$ only. Here $\boldsymbol{q}$ is the unit vector directed along the z axis (normal to the stratification planes). Let us present spatial dependence of the field vectors as

$$\{\boldsymbol{E}(\boldsymbol{r}), \boldsymbol{D}(\boldsymbol{r}), \boldsymbol{H}(\boldsymbol{r}), \boldsymbol{B}(\boldsymbol{r})\} = \{\boldsymbol{E}(z), \boldsymbol{D}(z), \boldsymbol{H}(z), \boldsymbol{B}(z)\}\exp(\mathrm{i}k\boldsymbol{b}\boldsymbol{r}) \tag{4}$$

where the unit vector $\boldsymbol{b}$ determines the propagation direction of the wave along the stratification planes, k being the wave number.

Then rotors $\nabla^\times$ in (1) can be replaced by the differential tensor operator $\boldsymbol{q}^\times \mathrm{d}/\mathrm{d}z + \mathrm{i}k\boldsymbol{b}^\times$. Taking into account (2) the wave equations are presented in the form

$$\frac{1}{\mathrm{i}k}\boldsymbol{q}^\times\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}z} = \nu\beta\boldsymbol{E} + \nu\mu\boldsymbol{H} - \boldsymbol{b}^\times\boldsymbol{E} \tag{5}$$

$$\frac{1}{\mathrm{i}k}\boldsymbol{q}^\times\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}z} = -\nu\varepsilon\boldsymbol{E} - \nu\alpha\boldsymbol{H} - \boldsymbol{b}^\times\boldsymbol{H} \tag{6}$$

where $\nu \equiv \omega/(ck)$ is the so-called dimensionless *reduced frequency* [12].

In equations (5) and (6) there are only two independent components of each of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$, and all the remaining components of these vectors can be expressed in terms of them. Let us transform these equations so that they contain only the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ (the projections of these vectors on the stratification planes). For this purpose we define a projective operator $I = 1 - \boldsymbol{q}\otimes\boldsymbol{q} = -\boldsymbol{q}^{\times 2}$ with the properties $I^2 = I$, $I\boldsymbol{q} = \boldsymbol{q}I = 0$, $I\boldsymbol{q}^\times = \boldsymbol{q}^\times I = \boldsymbol{q}^\times$, where 1 is a unit tensor in three-dimensional space. Then the tangential components of the vector fields are equal to $\boldsymbol{E}_\tau = I\boldsymbol{E}$, $\boldsymbol{H}_\tau = I\boldsymbol{H}$ and relations $\boldsymbol{E}_\tau = I\boldsymbol{E}_\tau$, $\boldsymbol{H}_\tau = I\boldsymbol{H}_\tau$ are valid. The decomposition of vector $\boldsymbol{H}$ into tangential and normal components is given by the formula

$$\boldsymbol{H} = (I + \boldsymbol{q}\otimes\boldsymbol{q})\boldsymbol{H} = \boldsymbol{H}_\tau + \boldsymbol{q}\cdot\boldsymbol{q}\boldsymbol{H}. \tag{7}$$

Similar decomposition exists for vector $\boldsymbol{E}$. The system equivalent to (5) and (6) is obtained as follows: the first equation is the sum of equations (5) and (6) multiplied from the left by tensor $\hat{\beta}$, and the second one is equation (6) multiplied from the left by tensor ε^{-1}. Taking into account (7) we have

$$\frac{1}{\mathrm{i}k}\frac{\mathrm{d}(\boldsymbol{q}\times\boldsymbol{E})}{\mathrm{d}z}+\frac{1}{\mathrm{i}k}\hat{\beta}\boldsymbol{q}^{\times}\frac{\mathrm{d}\boldsymbol{H}_{\tau}}{\mathrm{d}z}=\rho\boldsymbol{I}\boldsymbol{H}_{\tau}+\rho\boldsymbol{q}\cdot\boldsymbol{q}\boldsymbol{H}-\boldsymbol{b}^{\times}\boldsymbol{E} \tag{8}$$

$$\frac{1}{\mathrm{i}k}\varepsilon^{-1}\boldsymbol{q}^{\times}\frac{\mathrm{d}\boldsymbol{H}_{\tau}}{\mathrm{d}z}=-\tau\boldsymbol{I}\boldsymbol{H}_{\tau}-\tau\boldsymbol{q}\cdot\boldsymbol{q}\boldsymbol{H}-\nu\boldsymbol{E} \tag{9}$$

where

$$\rho=\nu\hat{\mu}-\hat{\beta}\boldsymbol{b}^{\times}\qquad\tau=\varepsilon^{-1}\boldsymbol{b}^{\times}+\nu\hat{\alpha}. \tag{10}$$

Now we multiply equation (8) from the left by vector $\boldsymbol{q}$, and equation (9) by vector $\boldsymbol{a}\equiv\boldsymbol{b}\times\boldsymbol{q}$

$$\frac{1}{\mathrm{i}k}\boldsymbol{q}\hat{\beta}\boldsymbol{q}^{\times}\frac{\mathrm{d}\boldsymbol{H}_{\tau}}{\mathrm{d}z}=\boldsymbol{q}\rho\boldsymbol{I}\boldsymbol{H}_{\tau}+\boldsymbol{q}\rho\boldsymbol{q}\cdot\boldsymbol{q}\boldsymbol{H}+\boldsymbol{a}\boldsymbol{E}$$

$$\frac{1}{\mathrm{i}k}\boldsymbol{a}\varepsilon^{-1}\boldsymbol{q}^{\times}\frac{\mathrm{d}\boldsymbol{H}_{\tau}}{\mathrm{d}z}=-\boldsymbol{a}\tau\boldsymbol{I}\boldsymbol{H}_{\tau}-\boldsymbol{a}\tau\boldsymbol{q}\cdot\boldsymbol{q}\boldsymbol{H}-\nu\boldsymbol{a}\boldsymbol{E}$$

whence it follows that the normal component of field $\boldsymbol{H}$ is equal to

$$\boldsymbol{q}\boldsymbol{H}=\frac{1}{\mathrm{i}kr}\boldsymbol{q}\theta\boldsymbol{q}^{\times}\frac{\mathrm{d}\boldsymbol{H}_{\tau}}{\mathrm{d}z}-\frac{1}{r}(\boldsymbol{a}\tau\boldsymbol{I}-\nu\boldsymbol{q}\rho\boldsymbol{I})\boldsymbol{H}_{\tau} \tag{11}$$

where

$$\theta=\boldsymbol{b}^{\times}\varepsilon^{-1}-\nu\hat{\beta} \tag{12}$$

$$r=\boldsymbol{a}\tau\boldsymbol{q}-\nu\boldsymbol{q}\rho\boldsymbol{q}=\boldsymbol{a}\varepsilon^{-1}\boldsymbol{a}+\nu\boldsymbol{a}\hat{\alpha}\boldsymbol{q}+\nu\boldsymbol{q}\hat{\beta}\boldsymbol{a}-\nu^{2}\boldsymbol{q}\hat{\mu}\boldsymbol{q}. \tag{13}$$

Scalar multiplying equation (9) by vector $\boldsymbol{q}$ and taking into consideration equation (11) gives an expression for the scalar product of vectors $\boldsymbol{q}$ and $\boldsymbol{\mathcal{E}}\equiv\nu\boldsymbol{E}$ as a function of $\boldsymbol{H}_{\tau}$ and $\mathrm{d}\boldsymbol{H}_{\tau}/\mathrm{d}z$:

$$\boldsymbol{q}\boldsymbol{\mathcal{E}}=-\frac{1}{\mathrm{i}k}\left(\boldsymbol{q}\varepsilon^{-1}\boldsymbol{q}^{\times}\boldsymbol{I}+\frac{1}{r}\boldsymbol{q}\tau\boldsymbol{q}\otimes\boldsymbol{q}\theta\boldsymbol{q}^{\times}\boldsymbol{I}\right)\frac{\mathrm{d}\boldsymbol{H}_{\tau}}{\mathrm{d}z}-\left[\boldsymbol{q}\tau\boldsymbol{I}-\frac{1}{r}\boldsymbol{q}\tau\boldsymbol{q}\otimes(\boldsymbol{a}\tau\boldsymbol{I}-\nu\boldsymbol{q}\rho\boldsymbol{I})\right]\boldsymbol{H}_{\tau}. \tag{14}$$

Vector product $\boldsymbol{q}\times\boldsymbol{\mathcal{E}}$ is found by multiplying equation (9) from the left by tensor $\boldsymbol{I}\boldsymbol{q}^{\times}$ and is represented by a formula similar to (14) with replacement of the factors $\boldsymbol{q}$ on the left in each member of (14) by $\boldsymbol{I}\boldsymbol{q}^{\times}$:

$$\begin{aligned}\boldsymbol{q}\times\boldsymbol{\mathcal{E}}=&-\frac{1}{\mathrm{i}k}\left(\boldsymbol{I}\boldsymbol{q}^{\times}\varepsilon^{-1}\boldsymbol{q}^{\times}\boldsymbol{I}+\frac{1}{r}\boldsymbol{I}\boldsymbol{q}^{\times}\tau\boldsymbol{q}\otimes\boldsymbol{q}\theta\boldsymbol{q}^{\times}\boldsymbol{I}\right)\frac{\mathrm{d}\boldsymbol{H}_{\tau}}{\mathrm{d}z}\\&-\left[\boldsymbol{I}\boldsymbol{q}^{\times}\tau\boldsymbol{I}-\frac{1}{r}\boldsymbol{I}\boldsymbol{q}^{\times}\tau\boldsymbol{q}\otimes(\boldsymbol{a}\tau\boldsymbol{I}-\nu\boldsymbol{q}\rho\boldsymbol{I})\right]\boldsymbol{H}_{\tau}.\end{aligned} \tag{15}$$

An additional equation that links vectors $\mathrm{d}(\boldsymbol{q}\times\boldsymbol{\mathcal{E}})/\mathrm{d}z$, $\mathrm{d}\boldsymbol{H}_{\tau}/\mathrm{d}z$ and $\boldsymbol{H}_{\tau}$ is obtained from (8) by multiplication from the left by tensor $\nu\boldsymbol{I}$:

$$\frac{1}{\mathrm{i}k}\frac{\mathrm{d}(\boldsymbol{q}\times\boldsymbol{\mathcal{E}})}{\mathrm{d}z}+\frac{\nu}{\mathrm{i}k}\boldsymbol{I}\hat{\beta}\boldsymbol{q}^{\times}\frac{\mathrm{d}\boldsymbol{H}_{\tau}}{\mathrm{d}z}=\nu\boldsymbol{I}\rho\boldsymbol{I}\boldsymbol{H}_{\tau}+\nu\boldsymbol{I}\rho\boldsymbol{q}\otimes\boldsymbol{q}\boldsymbol{H}-\boldsymbol{a}\otimes\boldsymbol{q}\boldsymbol{\mathcal{E}}. \tag{16}$$

In equation (16) it is taken into account that $\boldsymbol{I}\boldsymbol{b}^{\times}=-\boldsymbol{q}^{\times}(\boldsymbol{q}^{\times}\boldsymbol{b}^{\times})=-\boldsymbol{q}^{\times}\boldsymbol{b}\otimes\boldsymbol{q}=\boldsymbol{a}\otimes\boldsymbol{q}$. After substitution of (11) and (14) this equation takes the form

$$\begin{aligned}\frac{1}{\mathrm{i}k}\frac{\mathrm{d}(\boldsymbol{q}\times\boldsymbol{\mathcal{E}})}{\mathrm{d}z}=&\frac{1}{\mathrm{i}k}\left[\boldsymbol{I}\theta\boldsymbol{q}^{\times}\boldsymbol{I}+\frac{1}{r}(\boldsymbol{I}\boldsymbol{b}^{\times}\tau\boldsymbol{q}+\nu\boldsymbol{I}\rho\boldsymbol{q})\otimes\boldsymbol{q}\theta\boldsymbol{q}^{\times}\boldsymbol{I}\right]\frac{\mathrm{d}\boldsymbol{H}_{\tau}}{\mathrm{d}z}\\&+\left[\boldsymbol{I}\boldsymbol{b}^{\times}\tau\boldsymbol{I}+\nu\boldsymbol{I}\rho\boldsymbol{I}-\frac{1}{r}(\boldsymbol{I}\boldsymbol{b}^{\times}\tau\boldsymbol{q}+\nu\boldsymbol{I}\rho\boldsymbol{q})\otimes(\boldsymbol{a}\tau\boldsymbol{I}-\nu\boldsymbol{q}\rho\boldsymbol{I})\right]\boldsymbol{H}_{\tau}.\end{aligned} \tag{17}$$

Equations (15) and (17) for $\boldsymbol{H}_{\tau}(z)$, $\boldsymbol{q}\times\boldsymbol{\mathcal{E}}(z)$ are equivalent to basic equations (5) and (6) for $\boldsymbol{H}(z)$, $\boldsymbol{E}(z)$. After substitution of tensors ρ, τ (10), θ (12) and scalar quantity r (13)

coefficients before $\mathrm{d}\boldsymbol{H}_\tau/\mathrm{d}z$ and $\boldsymbol{H}_\tau$ in these equations are bulky. However equations (15) and (17) can be written in a more compact way if we use the following tensorial bilinear form $(\boldsymbol{uv})$ of arbitrary vector arguments $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$
\begin{aligned}
(\boldsymbol{uv}) = {} & \boldsymbol{I}\boldsymbol{u}^\times\varepsilon^{-1}\boldsymbol{v}^\times\boldsymbol{I} + \nu\boldsymbol{I}\boldsymbol{u}^\times\hat{\alpha}\boldsymbol{I}\cdot\boldsymbol{bv} - \nu\boldsymbol{bu}\cdot\boldsymbol{I}\hat{\beta}\boldsymbol{v}^\times\boldsymbol{I} + \nu^2\boldsymbol{bu}\cdot\boldsymbol{bv}\cdot\boldsymbol{I}\hat{\mu}\boldsymbol{I} \\
& - \frac{1}{r}(\boldsymbol{I}\boldsymbol{u}^\times\varepsilon^{-1}\boldsymbol{a} + \nu\boldsymbol{I}\boldsymbol{u}^\times\hat{\alpha}\boldsymbol{q} - \nu\boldsymbol{bu}\cdot\boldsymbol{I}\hat{\beta}\boldsymbol{a} + \nu^2\boldsymbol{bu}\cdot\boldsymbol{I}\hat{\mu}\boldsymbol{q}) \otimes (\boldsymbol{a}\varepsilon^{-1}\boldsymbol{v}^\times\boldsymbol{I} \\
& + \nu\boldsymbol{q}\hat{\beta}\boldsymbol{v}^\times\boldsymbol{I} + \nu\boldsymbol{a}\hat{\alpha}\boldsymbol{I}\cdot\boldsymbol{bv} - \nu^2\boldsymbol{q}\hat{\mu}\boldsymbol{I}\cdot\boldsymbol{bv}).
\end{aligned}
\tag{18}
$$

Then

$$
\boldsymbol{q}\times\boldsymbol{\mathcal{E}} = -\frac{1}{\mathrm{i}k}(\boldsymbol{qq})\frac{\mathrm{d}\boldsymbol{H}_\tau}{\mathrm{d}z} - (\boldsymbol{qb})\boldsymbol{H}_\tau \qquad \frac{1}{\mathrm{i}k}\frac{\mathrm{d}(\boldsymbol{q}\times\boldsymbol{\mathcal{E}})}{\mathrm{d}z} = \frac{1}{\mathrm{i}k}(\boldsymbol{bq})\frac{\mathrm{d}\boldsymbol{H}_\tau}{\mathrm{d}z} + (\boldsymbol{bb})\boldsymbol{H}_\tau. \tag{19}
$$

Tensors $(\boldsymbol{uv})$ (18) are planar, since $(\boldsymbol{uv})\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{uv}) = 0$. In orthonormal basis $\boldsymbol{b}, \boldsymbol{q}, \boldsymbol{a}$ they are represented by a 3×3 matrix of the type

$$
\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & 0 & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}.
$$

Note that for non-absorbing bianisotropic media the relation $(\boldsymbol{uv})^+ = (\boldsymbol{vu})$ is valid.

For the tensorial bilinear form of type (18), there is no inverse tensor in three-dimensional space; however, it can be inverted in a two-dimensional subspace orthogonal to $\boldsymbol{q}$. The corresponding operation is called pseudoinversion, and pseudoinverse tensor $(\boldsymbol{uv})^-$ is introduced according to formula $(\boldsymbol{uv})^-(\boldsymbol{uv}) = (\boldsymbol{uv})(\boldsymbol{uv})^- = \boldsymbol{I}$. The projective operator I is the operator of identical transformation in the two-dimensional subspace. Multiplying the first equation (19) from the left by $(\boldsymbol{qq})^-$ we find the derivative $\mathrm{d}\boldsymbol{H}_\tau/\mathrm{d}z$ and then exclude it from the second equation. As a result we can write the system of wave equations in the following block-matrix form:

$$
\frac{\mathrm{d}U(z)}{\mathrm{d}z} = \mathrm{i}kN(z)U(z) \qquad U = \begin{pmatrix} \boldsymbol{H}_\tau \\ \boldsymbol{q}\times\boldsymbol{\mathcal{E}} \end{pmatrix} \qquad N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \tag{20}
$$

where tensorial elements of a block 6×6 matrix N assume the form

$$
\begin{aligned}
& N_{11} = -(\boldsymbol{qq})^-(\boldsymbol{qb}) && N_{12} = -(\boldsymbol{qq})^- \\
& N_{21} = -[(\boldsymbol{bq})(\boldsymbol{qq})^-(\boldsymbol{qb}) - (\boldsymbol{bb})] && N_{22} = -(\boldsymbol{bq})(\boldsymbol{qq})^-.
\end{aligned}
\tag{21}
$$

In view of the fact that tensor (18) for non-absorbing bianisotropic media possesses the above-mentioned symmetry properties, it is not difficult to verify that for these media

$$
N_{12}^+ = N_{12} \qquad N_{21}^+ = N_{21} \qquad N_{11}^+ = N_{22}. \tag{22}
$$

Thus, system (20) of the first-order differential equations describes the propagation of monochromatic electromagnetic waves in stratified bianisotropic media. The fundamental solution of this system can be presented in an evolution form by means of a product integral [21]

$$
U(z) = \widehat{\int_{z_0}^{z}}(E + \mathrm{i}kN(z)\,\mathrm{d}z)U(z_0) \tag{23}
$$

where

$$
E = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{24}
$$

and $U(z_0)$ is a predetermined field vector in the plane $z = z_0$. In a general case for arbitrary profile of the inhomogeneity $\varepsilon = \varepsilon(z)$, $\mu = \mu(z)$, $\alpha = \alpha(z)$, $\beta = \beta(z)$ the solution of these systems can only be found numerically.

The system of equations (20) with matrix N, elements of which are introduced in the manner of (21), is used for different branches of physics. It describes motion of linear straight dislocations in solid bodies [9, 22] and propagation of elastic waves in stratified anisotropic media. In the theory of surface acoustic waves the wave equations in the form (20), (21) were first obtained by Stroh [16] and then used in the work of Ingebrigtsen and Tonning [23].

Thus, surface waves (acoustic and electromagnetic) are described by a system of equations (20). The representation of wave equations in the form (20), (21) gives rise to the integral approach of Barnett and Lothe [9, 10]. In the next sections we develop such an approach for surface polaritons in non-absorbing bianisotropic media taking into account planar structure of tensor $(\boldsymbol{u}\boldsymbol{v})$ (18) and symmetry properties (22) of the matrix elements N_{11}, N_{12}, N_{21}, N_{22}.

3. Surface electromagnetic waves at the interface of bianisotropic media. Dispersion equations

We consider propagation of surface electromagnetic waves with frequency ω along the plane interface of two non-absorbing bianisotropic media, characterized by tensors $\varepsilon(\omega) = \varepsilon^{+}(\omega)$, $\mu(\omega) = \mu^{+}(\omega), \alpha(\omega) = \beta^{+}(\omega)$ and $\varepsilon'(\omega) = \varepsilon'^{+}(\omega)$, $\mu'(\omega) = \mu'^{+}(\omega)$, $\alpha'(\omega) = \beta'^{+}(\omega)$, respectively. Suppose that principal axes of these tensors, determined by their complex eigenvectors, are arbitrarily oriented with respect to the interface. We align a coordinate plane $z = 0$ with the boundary between the media. Hereinafter all symbols with a prime refer to the medium with tensors ε', μ', α', β', which is located in a half-space $z > 0$. Field distribution in both media is represented by a superposition of two inhomogeneous partial waves. For instance in the medium $z < 0$, we have

$$\boldsymbol{H}(\boldsymbol{r}, t) = \sum_{s=1}^{2} C_s \boldsymbol{H}_s^0 \exp[\mathrm{i}k(\boldsymbol{b} + \eta_s \boldsymbol{q})\boldsymbol{r} - \mathrm{i}\omega t] = \sum_{s=1}^{2} C_s \boldsymbol{H}_s^0 \exp\left[\mathrm{i}\omega\left(\frac{1}{c}\boldsymbol{m}_s \boldsymbol{r} - t\right)\right] \qquad (25)$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = \sum_{s=1}^{2} C_s \boldsymbol{E}_s^0 \exp[\mathrm{i}k(\boldsymbol{b} + \eta_s \boldsymbol{q})\boldsymbol{r} - \mathrm{i}\omega t] = \sum_{s=1}^{2} C_s \boldsymbol{E}_s^0 \exp\left[\mathrm{i}\omega\left(\frac{1}{c}\boldsymbol{m}_s \boldsymbol{r} - t\right)\right] \qquad (26)$$

where as above $\boldsymbol{q}$ is the unit normal to the boundary directed along the z axis; a unit vector $\boldsymbol{b}$ ($\boldsymbol{b}\boldsymbol{q} = 0$) determines the propagation direction of the wave along the boundary; $\boldsymbol{H}_s^0$, $\boldsymbol{E}_s^0$ are the vector amplitudes of the partial waves and C_s are weight factors. Complex coefficients η_s characterize decay of the surface wave when moving away from the interface, and $\mathrm{Im}\,\eta_s < 0$. In (26) $\boldsymbol{m}_s$ are the complex refraction vectors [20] of inhomogeneous partial waves:

$$\boldsymbol{m}_s = \frac{1}{\nu}(\boldsymbol{b} + \eta_s \boldsymbol{q}) \qquad s = 1, 2 \qquad (27)$$

where reduced frequency $\nu = \omega/(ck)$ represents phase velocity of the surface wave in units of c (velocity of light in vacuum). Fields $\boldsymbol{H}'(\boldsymbol{r}, t)$, $\boldsymbol{E}'(\boldsymbol{r}, t)$ in the second medium ($z > 0$) are also described by equations of type (26) with the change of symbols $\boldsymbol{H}_s^0$, $\boldsymbol{E}_s^0$, C_s, η_s, $\boldsymbol{m}_s$ to the same with a prime. Here the decay coefficients η_s' are subject to the condition $\mathrm{Im}\,\eta_s' > 0$.

Quantities η_s are found from the Fresnel equation for bianisotropic media [20] $\det[\mu^{-1}(\boldsymbol{m}^{\times} - \alpha^{+})\varepsilon^{-1}(\boldsymbol{m}^{\times} + \alpha) + 1] = 0$ with the vector $\boldsymbol{m}$ substituted in form (27). Quantities η_s' are found from an analogous equation with tensors ε', μ', α'. This is a fourth-order algebraic equation with respect to η with coefficients depending on ν.

If parameter ν does not exceed a value ν_{L}, called the *limiting frequency* [12], then roots η_j ($j = 1, \ldots, 4$) of the Fresnel equation are pairs of complex conjugate numbers

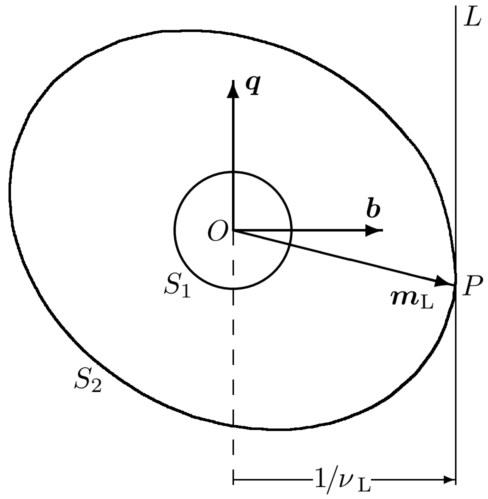

Figure 1. Determination of limiting frequency ν_L using the refraction surface section.

(in equation (26) we use only these with negative imaginary part). It is convenient to find the limiting frequency ν_L geometrically using the refraction surface section (the surface of the refraction indices for partial waves) by the plane passing through vectors $\boldsymbol{b}$ and $\boldsymbol{q}$. Let L be a straight line parallel to vector $\boldsymbol{q}$ and originally located at infinity, which moves towards $-\boldsymbol{b}$ until the first contact with section curve S_2 of the outer sheet of the refraction surface (figure 1). Then the limiting frequency ν_L is reciprocal to the distance from the reference point O to the line L, passing through point of tangency P. When $\nu = \nu_\mathrm{L}$ both roots η_i of the Fresnel equation become real (or all four roots if there are two points of tangency) and correspond to the body partial waves with the refraction vectors $\boldsymbol{m}_\mathrm{L}$.

Usually for derivation of the dispersion equations $k = k(\omega)$ for surface electromagnetic waves, it is necessary to first find the amplitudes of the partial waves and then use boundary conditions for tangential components of these amplitudes. In the case under consideration the amplitudes $\boldsymbol{H}_s^0$, $\boldsymbol{E}_s^0$ satisfy the equations

$$\left[\mu^{-1}\left(\boldsymbol{m}_s^\times - \alpha^+\right)\varepsilon^{-1}\left(\boldsymbol{m}_s^\times + \alpha\right) + 1\right]\boldsymbol{H}_s^0 = 0$$

$$\boldsymbol{E}_s^0 = -\varepsilon^{-1}\left(\boldsymbol{m}_s^\times - \alpha^+\right)\boldsymbol{H}_s^0 \qquad s = 1, 2$$

(similar equations are valid for $\boldsymbol{H}'^0_s$, $\boldsymbol{E}'^0_s$). Boundary conditions are written in the form

$$\boldsymbol{H}_\tau^0 = \boldsymbol{H}'^0_\tau \qquad \boldsymbol{q}\times\boldsymbol{E}^0 = \boldsymbol{q}\times\boldsymbol{E}'^0 \tag{28}$$

where

$$\begin{aligned} \boldsymbol{H}_\tau^0 &= \sum_{s=1}^{2} C_s \boldsymbol{H}_{s\tau}^0 \qquad & \boldsymbol{q}\times\boldsymbol{E}^0 &= \sum_{s=1}^{2} C_s \boldsymbol{q}\times\boldsymbol{E}_s^0 \\ \boldsymbol{H}'^0_\tau &= \sum_{s=1}^{2} C'_s \boldsymbol{H}'^0_{s\tau} \qquad & \boldsymbol{q}\times\boldsymbol{E}'^0 &= \sum_{s=1}^{2} C'_s \boldsymbol{q}\times\boldsymbol{E}'^0_s. \end{aligned} \tag{29}$$

From equations (28) and (29) factors C_s, C'_s can be excluded and as a result we get the dispersion equation in the form of $F(\nu) = 0$. Its solution $\nu = \nu_\mathrm{S}$ will describe the surface wave only if $0 < \nu_\mathrm{S} < \hat{\nu}_\mathrm{L}$, where $\hat{\nu}_\mathrm{L} = \min(\nu_\mathrm{L}, \nu'_\mathrm{L})$ is the least of the limiting frequencies for two contacting media.

A different way of obtaining the dispersion equation is by using the surface impedance tensors in the boundary conditions. Planar surface impedance tensors γ and γ' connect the tangential components of electric and magnetic fields on the boundary [18]:

$$\boldsymbol{q} \times \boldsymbol{E}^0 = \gamma \boldsymbol{H}^0_\tau \qquad \boldsymbol{q} \times \boldsymbol{E}'^0 = \gamma' \boldsymbol{H}'^0_\tau. \tag{30}$$

Eliminating vectors $\boldsymbol{H}'^0_\tau$, $\boldsymbol{q} \times \boldsymbol{E}^0$, $\boldsymbol{q} \times \boldsymbol{E}'^0$ from (28) and (30), we find

$$(\gamma - \gamma')\boldsymbol{H}^0_\tau = 0. \tag{31}$$

Non-zero vector $\boldsymbol{H}^0_\tau$ satisfies equation (31) only if the determinant of tensor $\gamma - \gamma'$ (considered as a tensor in two-dimensional space of the interface plane) vanishes. In a three-dimensional space this corresponds to vanishing of the trace of the tensor adjoined to $\gamma - \gamma'$ [20]:

$$\overline{(\gamma - \gamma')}_t = 0. \tag{32}$$

Tensors γ and γ' depend on ν, and relation (32) is the dispersion equation for the surface waves. In the next section a derivation of the surface impedance tensors γ and γ' based on the representation of the wave equations in the form (20), (21) is given.

4. Integral representation of surface impedance tensors

We introduce vectors

$$U_j = \begin{pmatrix} \boldsymbol{H}^0_{j\tau} \\ \boldsymbol{q} \times \boldsymbol{\mathcal{E}}^0_j \end{pmatrix}$$

constructed from the tangential amplitude components of partial waves in the medium $z < 0$. These vectors correspond to the complex-conjugate pairs of decay coefficients $\eta_j = \eta^*_{5-j}$, $j = 1, \ldots, 4$ ($0 \leqslant \nu < \nu_{\mathrm{L}}$) and belong to the four-dimensional *amplitude space*, being a direct sum of the two-dimensional spaces orthogonal to the vector $\boldsymbol{q}$. Taking into account (26), system (20) takes the form

$$NU_j = \eta_j U_j \tag{33}$$

where matrix N does not depend on z and is a function of tensors ε, μ, α and β. Thus, U_j are right eigenvectors of the matrix N in the amplitude space, and η_j are corresponding eigenvalues. In view of relations (22), we find that

$$TN = N^+ T \tag{34}$$

where

$$T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Now introduce vectors $W_j = TU_{5-j}$, $j = 1, \ldots, 4$. From (33) and (34) it follows that W^+_j are left eigenvectors of the matrix N:

$$W^+_j N = \eta_j W^+_j.$$

If all the eigenvalues η_j are different, then vectors U_j and W^+_i ($i \neq j$) are biorthogonal: $W^+_i U_j = 0$. These vectors can be normalized

$$W^+_i U_j = \delta_{ij} \qquad i, j = 1, \ldots, 4 \tag{35}$$

where δ_{ij} is a Kronecker delta. Thus, vectors U_j form a basis in the amplitude space, and vectors W_j form the adjoined basis. The completeness condition is written in the form

$$\sum_{j=1}^{4} U_j \otimes W^+_j = E \tag{36}$$

where E (24) is an identical operator in the amplitude space. For tangential amplitude components of the partial waves, equations (35) and (36) take the form

$$\boldsymbol{H}^{0}_{j\tau}\left(\boldsymbol{q}\times\boldsymbol{\mathcal{E}}^{0*}_{5-i}\right)+\boldsymbol{H}^{0*}_{5-i\tau}\left(\boldsymbol{q}\times\boldsymbol{\mathcal{E}}^{0}_{j}\right)=\delta_{ij}\qquad i,j=1,\ldots,4$$

and

$$\sum_{j=1}^{4}\boldsymbol{H}^{0}_{j\tau}\otimes\left(\boldsymbol{q}\times\boldsymbol{\mathcal{E}}^{0*}_{5-j}\right)=I\qquad\sum_{j=1}^{4}\boldsymbol{H}^{0}_{j\tau}\otimes\boldsymbol{H}^{0*}_{5-j\tau}=0$$

$$\sum_{j=1}^{4}\left(\boldsymbol{q}\times\boldsymbol{\mathcal{E}}^{0}_{j}\right)\otimes\left(\boldsymbol{q}\times\boldsymbol{\mathcal{E}}^{0*}_{5-j}\right)=0.$$

Now consider matrix

$$N(\phi)=-\begin{pmatrix}(\boldsymbol{e}_2\boldsymbol{e}_2)^{-}(\boldsymbol{e}_2\boldsymbol{e}_1) & (\boldsymbol{e}_2\boldsymbol{e}_2)^{-}\\(\boldsymbol{e}_1\boldsymbol{e}_2)(\boldsymbol{e}_2\boldsymbol{e}_2)^{-}(\boldsymbol{e}_2\boldsymbol{e}_1)-(\boldsymbol{e}_1\boldsymbol{e}_1) & (\boldsymbol{e}_1\boldsymbol{e}_2)(\boldsymbol{e}_2\boldsymbol{e}_2)^{-}\end{pmatrix}$$

where elements are tensors (18) with vector arguments

$$\boldsymbol{e}_1=\boldsymbol{b}\cos\phi+\boldsymbol{q}\sin\phi\qquad\boldsymbol{e}_2=-\boldsymbol{b}\sin\phi+\boldsymbol{q}\cos\phi.\tag{37}$$

Its eigenvectors and eigenvalues depend on ϕ:

$$N(\phi)U_j(\phi)=\eta_j(\phi)U_j(\phi)\qquad W_j^{+}(\phi)N(\phi)=\eta_j(\phi)W_j^{+}(\phi)\tag{38}$$

and $W_i^{+}(\phi)U_j(\phi)=\delta_{ij}$. At $\phi=0$ matrix $N(\phi)$ coincides with the matrix N of system (20), and its eigenvectors and eigenvalues coincide with the quantities U_j, W_j and η_j which describe real physical fields. It is important that

$$\frac{\mathrm{d}N(\phi)}{\mathrm{d}\phi}=-\{E+[N(\phi)]^2\}.\tag{39}$$

Formula (39) is checked directly by calculation and comparison of the expressions on its left-hand and right-hand sides. Note that to calculate the derivative $\mathrm{d}N(\phi)/\mathrm{d}\phi$, we take into account relations

$$\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}\phi}=\boldsymbol{e}_2\qquad\frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}\phi}=-\boldsymbol{e}_1\qquad\frac{\mathrm{d}(\boldsymbol{u}\boldsymbol{v})}{\mathrm{d}\phi}=\left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\phi}\boldsymbol{v}\right)+\left(\boldsymbol{u}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\phi}\right)$$

$$\frac{\mathrm{d}(\boldsymbol{e}_2\boldsymbol{e}_2)^{-}}{\mathrm{d}\phi}=-(\boldsymbol{e}_2\boldsymbol{e}_2)^{-}\frac{\mathrm{d}(\boldsymbol{e}_2\boldsymbol{e}_2)}{\mathrm{d}\phi}(\boldsymbol{e}_2\boldsymbol{e}_2)^{-}.$$

The last of these equations can be obtained by differentiation of the equation $(\boldsymbol{e}_2\boldsymbol{e}_2)(\boldsymbol{e}_2\boldsymbol{e}_2)^{-}=I$ with respect to ϕ .

Differentiating both parts of the first equation (38) with respect to ϕ and taking into account (39) gives

$$[N(\phi)-\eta_j(\phi)E]\frac{\mathrm{d}U_j(\phi)}{\mathrm{d}\phi}=\left[\frac{\mathrm{d}\eta_j(\phi)}{\mathrm{d}\phi}+1+\eta_j^2(\phi)\right]U_j(\phi).\tag{40}$$

Then multiplying equation (40) from the left by vector $W_j^{+}(\phi)$, we obtain

$$\frac{\mathrm{d}\eta_j(\phi)}{\mathrm{d}\phi}=-1-\eta_j^2(\phi)\tag{41}$$

$$N(\phi)\frac{\mathrm{d}U_j(\phi)}{\mathrm{d}\phi}=\eta_j(\phi)\frac{\mathrm{d}U_j(\phi)}{\mathrm{d}\phi}.\tag{42}$$

It is obvious that $\mathrm{d}U_j(\phi)/\mathrm{d}\phi$ is a right eigenvector of the matrix $N(\phi)$ with the same eigenvalue as that for eigenvector $U_j(\phi)$. Suppose that all eigenvalues $\eta_j(\phi)$ are different and eigenvectors

$U_j(\phi)$ generate one-dimensional eigensubspaces (there is no degeneracy of eigenvalues), we have $\mathrm{d}U_j(\phi)/\mathrm{d}\phi = f_j(\phi)U_j(\phi)$, where $f_j(\phi)$ are some scalar functions. Thus when ϕ is changed, vectors $U_j(\phi)$ preserve their 'direction'. Since they are normalized, they also preserve their 'length', i.e. they do not depend on ϕ. So in equations (38) it is possible to replace $U_j(\phi)$, $W_j(\phi)$ by vectors of the partial wave amplitudes $U_j = U_j(0)$, $W_j = W_j(0)$ and then average over ϕ

$$\mathcal{N}U_j = p_j U_j \qquad \mathcal{N} = \begin{pmatrix} S & Q \\ B & S^+ \end{pmatrix} \tag{43}$$

where $p_j = \dfrac{1}{\pi}\displaystyle\int_0^\pi \eta_j(\phi)\,\mathrm{d}\phi$, and elements of matrix $\mathcal{N}$ equal

$$S = -\frac{1}{\pi}\int_0^\pi (e_2e_2)^-(e_2e_1)\,\mathrm{d}\phi \qquad Q = -\frac{1}{\pi}\int_0^\pi (e_2e_2)^-\,\mathrm{d}\phi \tag{44}$$

$$B = -\frac{1}{\pi}\int_0^\pi [(e_1e_2)(e_2e_2)^-(e_2e_1) - (e_1e_1)]\,\mathrm{d}\phi. \tag{45}$$

Solving differential equations (41) under initial conditions $\eta_j(0) = \eta_j$, we find that values p_j equal $-\mathrm{i}$ or i depending on the sign of the imaginary part of η_j (negative or positive). Equation (26) includes decay coefficients η_1, η_2 with negative imaginary part, so we have

$$\mathcal{N}U_{1,2} = -\mathrm{i}U_{1,2} \qquad \mathcal{N}U_{3,4} = \mathrm{i}U_{3,4}. \tag{46}$$

The first equation (46) is satisfied not only by the vectors U_1 and U_2 (amplitudes of the partial waves), but also by any linear combination of these vectors. Taking into consideration (29) and (30), we find

$$\begin{pmatrix} S & Q \\ B & S^+ \end{pmatrix}\begin{pmatrix} \boldsymbol{H}_\tau^0 \\ \nu\gamma\boldsymbol{H}_\tau^0 \end{pmatrix} = -\mathrm{i}\begin{pmatrix} \boldsymbol{H}_\tau^0 \\ \nu\gamma\boldsymbol{H}_\tau^0 \end{pmatrix} \tag{47}$$

whence

$$(\mathrm{i}I + S + \nu Q\gamma)\boldsymbol{H}_\tau^0 = 0. \tag{48}$$

Formula (48) is derived from the system of wave equations (33) under the condition $0 \leqslant \nu < \nu_\mathrm{L}$ and boundary conditions have not been used for its derivation. Consequently, equation (48) is valid for any predetermined tangential component $\boldsymbol{H}_\tau$ of the magnetic field on the boundary. This means that $\mathrm{i}I + S + \nu Q\gamma = 0$ and the surface impedance tensor equals

$$\gamma = \frac{1}{\nu}Q^-(-\mathrm{i}I - S). \tag{49}$$

Equations (47)–(49) were obtained under the assumption that eigenvalues η_j of the matrix N are different. It is possible to show that these equations are also true in the presence of the degeneracy ($\eta_1 = \eta_2$).

The surface impedance tensor γ' for the second medium $z > 0$ can be calculated similarly as

$$\gamma' = \frac{1}{\nu}Q'^-(\mathrm{i}I - S') \tag{50}$$

under the condition $0 \leqslant \nu < \nu'_\mathrm{L}$. Here, unlike (49), the projective tensor I is multiplied by +i since $\mathrm{Im}\,\eta'_{1,2} > 0$. Tensors S' and Q' have an integral representation (44) with the bilinear tensorial forms (18) where $\varepsilon \longrightarrow \varepsilon'$, $\mu \longrightarrow \mu'$, $\alpha \longrightarrow \alpha'$, $\beta \longrightarrow \beta'$.

Let us multiply equations (46) from the left by the matrix $\mathcal{N}$

$$\mathcal{N}^2U_j = -U_j \qquad j = 1, \ldots, 4. \tag{51}$$

Vectors U_1, U_2, U_3, U_4 form a basis in the amplitude space, and from (51) it follows that matrix $\mathcal{N}^2$ coincides with $-E$. Using representations (24) and (43) for matrices E and $\mathcal{N}$, we conclude that S, Q and B satisfy tensor equations

$$S^2 + QB + I = 0 \qquad BS + S^+B = 0 \qquad SQ + QS^+ = 0.$$

The last of these equations multiplied from the left and from the right by Q^- takes the form $Q^-S + S^+Q^- = 0$. Therefore, it is easy to check that the surface impedance tensor γ (49) is anti-Hermitian $\gamma^+ = -\gamma$ (it is also true for tensor γ' (50)). The anti-Hermitian property of tensors γ and γ' indicates that in any point of contacting media, the time averaged Poynting vector $\boldsymbol{S}$ is parallel to the boundary. Truly, tensor γ connects not only tangential field component $\boldsymbol{H}_\tau^0$ and $\boldsymbol{q} \times \boldsymbol{E}^0$ at the interface, but also tangential component $\boldsymbol{H}_\tau(z)$ and $\boldsymbol{q} \times \boldsymbol{E}(z)$ for any point $z = \boldsymbol{qr} < 0$, since the nature of dependences $\boldsymbol{H}_\tau(z)$ and $\boldsymbol{q} \times \boldsymbol{E}(z)$ on z is the same (see (26)). As a result it appears that energy flow of the wave is absent towards $\boldsymbol{q}$:

$$\begin{aligned} \boldsymbol{qS} &= \frac{c}{16\pi}\boldsymbol{q}[\boldsymbol{E}(z) \times \boldsymbol{H}^*(z) + \boldsymbol{E}^*(z) \times \boldsymbol{H}(z)] \\ &= \frac{c}{16\pi}[\boldsymbol{H}_\tau^*(z)(\boldsymbol{q} \times \boldsymbol{E}(z)) + (\boldsymbol{q} \times \boldsymbol{E}^*(z))\boldsymbol{H}_\tau(z)] \\ &= \frac{c}{16\pi}\boldsymbol{H}_\tau^*(z)(\gamma + \gamma^+)\boldsymbol{H}_\tau(z) = 0. \end{aligned}$$

5. Surface electromagnetic waves at the interface of bianisotropic and isotropic media

As an illustrative example of the method presented above, we consider the interface of bianisotropic and isotropic media and investigate the existence conditions of the surface electromagnetic waves.

Let an isotropic medium with inverse dielectric permittivity tensor $\varepsilon'^{-1} = a'1$ (where a' is inverse scalar permittivity) occupy upper half-space $z > 0$, and a bianisotropic medium characterized by material tensors

$$\varepsilon^{-1} = a1 + (b - a)\boldsymbol{q} \otimes \boldsymbol{q} \qquad \mu = \mu 1 \qquad \alpha = \beta = ig\boldsymbol{q}^\times \tag{52}$$

where $a = 1/\varepsilon_\perp$, $b = 1/\varepsilon_{||}$, occupies lower half-space $z < 0$. Medium (52) corresponds to the crystals of symmetry classes 3 m, 4 mm and 6 mm. We consider the case of optical axis perpendicular to the boundary. Then tensors $(\boldsymbol{e}_2\boldsymbol{e}_2)$, $(\boldsymbol{e}_1\boldsymbol{e}_1)$ take the form

$$\begin{aligned} (\boldsymbol{e}_2\boldsymbol{e}_1) = {} & \{-\mathrm{i}ag\nu + [b - a - (\mu - ag^2)\nu^2]\cos\phi\sin\phi\}\boldsymbol{a} \otimes \boldsymbol{a} \\ & + \frac{\mu\nu^2}{a - \mu\nu^2}[\mathrm{i}ag\nu + (\mu - ag^2)\nu^2\cos\phi\sin\phi]\boldsymbol{b} \otimes \boldsymbol{b} \end{aligned} \tag{53}$$

$$\begin{aligned} (\boldsymbol{e}_2\boldsymbol{e}_2) = {} & \{-a\cos^2\phi - [b - (\mu - ag^2)\nu^2]\sin^2\phi\}\boldsymbol{a} \otimes \boldsymbol{a} \\ & + \frac{\mu\nu^2}{a - \mu\nu^2}\{a\cos^2\phi + [a - (\mu - ag^2)\nu^2]\sin^2\phi\}\boldsymbol{b} \otimes \boldsymbol{b}. \end{aligned} \tag{54}$$

Now we calculate pseudoinverse tensor $(\boldsymbol{e}_2\boldsymbol{e}_2)^-$:

$$\begin{aligned} (\boldsymbol{e}_2\boldsymbol{e}_2)^- = {} & \frac{-1}{a\cos^2\phi + [b - (\mu - ag^2)\nu^2]\sin^2\phi}\boldsymbol{a} \otimes \boldsymbol{a} \\ & + \frac{a - \mu\nu^2}{\mu\nu^2\{a\cos^2\phi + [a - (\mu - ag^2)\nu^2]\sin^2\phi\}}\boldsymbol{b} \otimes \boldsymbol{b} \end{aligned} \tag{55}$$

and product of tensors $(\boldsymbol{e}_2\boldsymbol{e}_2)^-$ and $(\boldsymbol{e}_2\boldsymbol{e}_1)$:

$$(\boldsymbol{e}_2\boldsymbol{e}_2)^-(\boldsymbol{e}_2\boldsymbol{e}_1) = \frac{\mathrm{i}ag\nu - [b - a - (\mu - ag^2)\nu^2]\cos\phi\sin\phi}{a\cos^2\phi + [b - (\mu - ag^2)\nu^2]\sin^2\phi}\boldsymbol{a}\otimes\boldsymbol{a} + \frac{\mathrm{i}ag\nu + (\mu - ag^2)\nu^2\cos\phi\sin\phi}{a\cos^2\phi + [a - (\mu - ag^2)\nu^2]\sin^2\phi}\boldsymbol{b}\otimes\boldsymbol{b}. \tag{56}$$

Clearly, tensors Q and S (44) are expressed in terms of integrals

$$J_{(1;2)} = \frac{1}{\pi}\int_0^\pi \frac{(1; \cos\phi\sin\phi)\,\mathrm{d}\phi}{k_1\cos^2\phi + k_2\sin^2\phi}. \tag{57}$$

Integral J_2 equals zero and $J_1 = 1/(\sqrt{k_1k_2})$. Next, substituting formulae (55) and (56) into (44) yields expressions for Q and S:

$$Q = \frac{1}{\sqrt{a[b - (\mu - ag^2)\nu^2]}}\boldsymbol{a}\otimes\boldsymbol{a} - \frac{a - \mu\nu^2}{\mu\nu^2\sqrt{a[a - (\mu - ag^2)\nu^2]}}\boldsymbol{b}\otimes\boldsymbol{b} \tag{58}$$

$$S = \frac{-\mathrm{i}ag\nu}{\sqrt{a[b - (\mu - ag^2)\nu^2]}}\boldsymbol{a}\otimes\boldsymbol{a} + \frac{-\mathrm{i}ag\nu}{\sqrt{a[a - (\mu - ag^2)\nu^2]}}\boldsymbol{b}\otimes\boldsymbol{b}. \tag{59}$$

Finally from formulae (49), (58) and (59) we arrive at the following expression for the surface impedance tensor

$$\gamma = \frac{\mathrm{i}\mu\nu}{a - \mu\nu^2}\left\{\sqrt{a[a - (\mu - ag^2)\nu^2]} - ag\nu\right\}\boldsymbol{b}\otimes\boldsymbol{b} - \frac{\mathrm{i}}{\nu}\left\{\sqrt{a[b - (\mu - ag^2)\nu^2]} - ag\nu\right\}\boldsymbol{a}\otimes\boldsymbol{a}. \tag{60}$$

Similarly it is possible to find γ' (50) for the isotropic medium occupying the upper half-space

$$\gamma' = -\mathrm{i}\nu\sqrt{\frac{a'}{a' - \nu^2}}\boldsymbol{b}\otimes\boldsymbol{b} + \frac{\mathrm{i}}{\nu}\sqrt{a'(a' - \nu^2)}\boldsymbol{a}\otimes\boldsymbol{a}. \tag{61}$$

Using surface impedance tensors (60) and (61), we get the dispersion equation:

$$F(\nu) = 0 \tag{62}$$

$$F(\nu) = \left\{\frac{\mu}{a - \mu\nu^2}\left[\sqrt{a[a - (\mu - ag^2)\nu^2]} - ag\nu\right] + \sqrt{\frac{a'}{a' - \nu^2}}\right\} \times\left\{\sqrt{a[b - (\mu - ag^2)\nu^2]} - ag\nu + \sqrt{a'(a' - \nu^2)}\right\}. \tag{63}$$

Solutions of the dispersion equation (62) exist if some conditions are valid.

Namely, for a positive crystal $a > b$, we have

- for given a, b, g parameter a' must satisfy the condition

$$b/\mu < a' \leqslant \frac{b + \sqrt{b[b - 4a^2g^2(\mu - ag^2)]}}{2(\mu - ag^2)} \tag{64}$$

- for given $a, b, a'(a' > b/\mu')$ parameter g must satisfy the condition

$$\frac{a'(\mu a' - b)}{a(ab + a'^2)} \leqslant g^2 < \mu/a. \tag{65}$$

For a negative crystal $a < b$, we have more complicated conditions:

- for given a, b, g parameter a' must satisfy the condition

$$b/\mu < a' \leqslant \frac{a + \sqrt{A_1}}{2(\mu - ag^2)} \tag{66}$$

$$A_1 = a^2 - 8a^2 g\sqrt{b-a}(\mu - ag^2) + 4a(\mu - ag^2)[ag^2(2a - b) + \mu(b - a)] \tag{67}$$

whence it is possible to obtain the condition for g.

When condition (64), (65) or (66) is satisfied, there exists an exact solution of the dispersion equation (62)

$$\nu^2 = \frac{(a'^2 - ab)(a' - a\mu) + 2a^2 a'^2 g + 2ag\sqrt{aa'}\sqrt{ag^2 a'^3 - (a'^2 - ab)(a'\mu - b)}}{(a' - a\mu)^2 + 4a^2 a' g^2}. \tag{68}$$

For a natural material, the parameter g is much less than any component of the dielectric permeability tensor. To satisfy the existence conditions of surface waves (64), (65) or (66), it is necessary to choose contacting media so that b/μ and a' should differ to a small extent. For example, these media may be crystalline and amorphous forms of the same material, the first possessing gyrotropic properties, but the second possessing none. Thus, their dielectric permeabilities differ very little.

6. Conclusion

Analogously to anisotropic dielectric media [11, 12], the dispersion equation for surface polaritons at the interface of bianisotropic media can be derived with the use of integral representation of tensors Q, S (44) involved in the surface impedance tensors γ, γ' (49) and (50). So we have a general method for analytical derivation of the dispersion equations for bianisotropic media of any symmetry classes, the interface arbitrarily orientated with respect to crystallographic axes. This method is essentially a consecutive calculation of tensor $(\boldsymbol{e}_2\boldsymbol{e}_2)$, $(\boldsymbol{e}_2\boldsymbol{e}_1)$ (18) and (37) and then Q, S, γ (44) and (49) for each contacting medium.

Convenience of the proposed method lies in independent calculation of the surface impedance tensors γ, γ' for each medium. Only at the last stage when these tensors are substituted into equation (32) will the final dispersion equation be obtained. Thus, having the surface impedance tensors (49) and (50) for n bianisotropic media (which may differ in symmetry classes and/or in orientation of the plane boundary with respect to crystallographic axes), we can immediately obtain $n(n-1)/2$ dispersion equations for each pair of such media.

In this paper we have derived the surface impedance tensor for bianisotropic media of 3 m, 4 mm and 6 mm symmetry classes in the case when the optical axis is perpendicular to the boundary plane. We have obtained the corresponding dispersion equation for the boundary of such media with isotropic ones. It is evident that analogous, if somewhat more complicated, consideration can be carried out for the same materials but with optical axis arbitrarily oriented with respect to the interface.

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